Physica D 238 (2009) 2062–2066

Contents lists available at ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

An improved local blow-up condition for Euler–Poisson equations with attractive forcing

Bin Cheng [a], Eitan Tadmor [b,*]

[a] Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA

[b] Department of Mathematics, Institute for Physical Science and Technology and Center of Scientific Computation And Mathematical Modeling (CSCAMM), University of Maryland, College Park, MD 20742, USA

ARTICLE INFO

Article history:
Received 10 February 2009
Received in revised form
28 July 2009
Accepted 5 August 2009
Available online 19 August 2009
Communicated by K. Promislow

Keywords:
Euler–Poisson equations
Critical thresholds
Finite time blow-up

ABSTRACT

We improve the recent result of Chae and Tadmor (2008) [10] proving a one-sided threshold condition which leads to a finite-time breakdown of the Euler–Poisson equations in arbitrary dimension n.

1. Introduction

The pressure-less Euler–Poisson (EP) equations in dimension $n \geq 1$ are

$$\rho_t + \operatorname{div}(\rho \mathbf{u}) = 0 \tag{1.1a}$$

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = k \nabla \Delta^{-1}(\rho - c), \tag{1.1b}$$

governing the unknown density $\rho = \rho(t, x) : \mathbb{R}_+ \times \mathbb{R}^n \mapsto \mathbb{R}_+$ and velocity $\mathbf{u} = \mathbf{u}(t, x) : \mathbb{R}_+ \times \mathbb{R}^n \mapsto \mathbb{R}^n$ subject to initial conditions $\rho(0, x) = \rho_0(x)$ and $\mathbf{u}(0, x) = \mathbf{u}_0(x)$. They involve two constants: (i) a fixed background state $c \geq 0$ — typical cases include the case of zero background, $c = 0$, or the case of a nonzero background given by the average mass, $c = \int \rho(t, x) \mathrm{d}x = \int \rho_0(x) \mathrm{d}x$; and (ii) a constant k which parameterizes the repulsive $k > 0$ or attractive $k < 0$ forcing, governed by the Poisson potential $\Delta^{-1}(\rho - c)$. The EP system appears in numerous applications including semiconductors and plasma physics ($k > 0$) and the collapse of stars due to self gravitation ($k < 0$) [1–4]. In particular, the pressureless EP model becomes relevant in interstellar clouds where gravitional forces dominate pressure gradient, [5], for example, or in the context of the Euler–Monge–Ampère systems and their quasi-neutral limits to the *incompressible* Euler equations [6].

This paper is restricted to the *attractive case*, $k < 0$. We begin by setting $c = 1$, $k = -1$ in (1.1a), (1.1b) to arrive at the unit-free EP system,

$$\rho_t + \operatorname{div}(\rho \mathbf{u}) = 0, \tag{1.2a}$$

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla \Delta^{-1}(\rho - 1). \tag{1.2b}$$

Our discussion remains valid for the general physical parameters $c \geq 0$, $k < 0$ upon a simple rescaling and limiting arguments, outlined in Corollary 1.1 below for $c > 0$ and Corollary 1.2 for the case of zero background $c = 0$.

We are concerned here with the persistence of C^1 regularity for solutions of the attractive EP system. Our Main theorem reveals a *pointwise* criterion on the initial data, a so-called critical threshold criterion [7–9], that leads to finite time blow-up of $\nabla \mathbf{u}$. It quantifies the balance between the two term $\operatorname{div} \mathbf{u}$ and ρ, which govern two competing mechanisms that dictate the C^1 regularity of EP flows. Our result also stands out as a generalization of several existing results [7,10,11,9] for which further discussion is given after the Main theorem and its corollary.

Main Theorem 1.1. *Consider the n-dimensional, attractive Euler–Poisson system (1.2a), (1.2b) subject to initial data ρ_0, $\mathbf{u}_0$. Then, the solution will lose C^1 regularity at a finite time $t = t_c < \infty$, if there exists a non-vacuum initial state $\rho_0(\bar{x}) > 0$ with vanishing initial*

* Corresponding author.
E-mail addresses: bincheng@umich.edu (B. Cheng), tadmor@cscamm.umd.edu (E. Tadmor).
URLs: http://www.umich.edu/~bincheng (B. Cheng), http://www.cscamm.umd.edu/~tadmor (E. Tadmor).

0167-2789/$ – see front matter
doi:10.1016/j.physd.2009.08.008

vorticity, $\nabla\times\mathbf{u}_0(\bar{x}) = 0$*, at some* $\bar{x}\in\mathbb{R}^n$ *such that the following sup-critical condition is fulfilled,*

$$\operatorname{div}\mathbf{u}_0(\bar{x}) < \operatorname{sgn}(\rho_0(\bar{x})-1)\sqrt{nF(\rho_0(\bar{x}))}, \tag{1.3a}$$

where

$$F(\rho) := \begin{cases} 1+\dfrac{2\rho}{n-2}-\dfrac{n\rho^{2/n}}{n-2}, & n\neq 2,\\ 1-\rho+\rho\ln\rho, & n=2. \end{cases} \tag{1.3b}$$

In particular, $\min_x \operatorname{div}\mathbf{u}(t,x)\to-\infty$ *and* $\max_x\rho(t,x)\to\infty$ *as* $t\uparrow t_c$.

Proof. Combine Lemmas 3.1 and 4.2, while noting that the curve

$$\operatorname{div}\mathbf{u} = \operatorname{sgn}(\rho-1)\sqrt{nF(\rho)},$$

is the separatrix along the boundary of the blow-up region $\Omega = \Omega_1\cup\Omega_2$ defined in (4.3) and illustrated in Fig. 4.1. □

We note in passing that, by classical arguments, the force-free Euler system $\mathbf{u}_t+\mathbf{u}\cdot\nabla\mathbf{u} = 0$ exhibits finite time blow-up if and only if there exists at least one *negative* eigenvalue of $\nabla\mathbf{u}_0(\bar{x})$. In the above theorem, however, finite-time blow-up can occur solely depending on the initial profile of $\operatorname{div}\mathbf{u}_0$ and ρ_0 regardless of individual eigenvalues of $\nabla\mathbf{u}_0$.

We also note that, by rescaling ρ to ρ/c, x to $\sqrt{-kc}\,x$ and t to $\sqrt{-kc}\,t$, the Main theorem immediately applies to the EP system (1.1a), (1.1b) with physical parameters. Since the EP system with $k<0$ models the collapse of interstellar cloud, the following corollary reveals a pointwise condition for mass concentration, $\rho\to\infty$, which interestingly preludes the birth of new stars.

Corollary 1.1. *Consider the Euler–Poisson system (1.1a), (1.1b) with* $c>0$, $k<0$ *subject to initial data* ρ_0, $\mathbf{u}_0$*. Then, the solution will lose* C^1 *regularity at a finite time* $t_c<\infty$*, if there exists a non-vacuum initial state* $\rho_0(\bar{x})>0$ *with a vanishing initial vorticity,* $\nabla\times\mathbf{u}_0(\bar{x}) = 0$*, such that the super-critical condition is fulfilled,*

$$\operatorname{div}\mathbf{u}_0(\bar{x}) < \operatorname{sgn}(\rho_0(\bar{x})-c)\sqrt{-nkcF\left(\frac{\rho_0(\bar{x})}{c}\right)} \tag{1.4}$$

where $F(\cdot)$ *is given in (1.3b). In particular,* $\min_x\operatorname{div}\mathbf{u}(t,x)\to-\infty$ *and* $\max_x\rho(t,x)\to\infty$ *as* $t\uparrow t_c$.

In the limiting regime as $c\to 0+$, condition (1.4) converges to a super-critical condition which is summarized by the following result, the proof of which is given in Section 5.

Corollary 1.2. *Consider the n-dimensional Euler–Poisson system (1.1a), (1.1b) with* $c=0$, $k<0$ *subject to initial data* ρ_0, $\mathbf{u}_0$*. Assume a vanishing initial vorticity everywhere,* $\nabla\times\mathbf{u}_0\equiv 0$*. Then, the solution will lose* C^1 *regularity at a finite time* $t_c<\infty$*, if either* (i) $n=1,2$ *or* (ii) $n\geq 3$ *and there exists a non-vacuum initial state* $\rho_0(\bar{x})>0$ *such that*

$$\operatorname{div}\mathbf{u}_0(\bar{x}) < \sqrt{-\frac{2nk\rho_0(\bar{x})}{n-2}},\quad n\geq 3. \tag{1.5}$$

In other words, the pressureless and vorticity-free one- and two-dimensional attractive Euler–Poisson systems with zero background ($c=0$), inevitably collapse to singularity at a finite time. On the other hand, the complete characterization of finite-time breakdown in higher dimensions remains open, even for $c=0$.

The concept of Critical Threshold and associated methodology is originated and developed in a series of papers by Engelberg, Liu and Tadmor [7], Liu and Tadmor [9,8] and more. It first appears in [7] regarding pointwise criteria for C^1 solution regularity of 1D EP system. The key argument in that paper is based on the convective derivative along particle paths $' = \partial_t+\mathbf{u}\cdot\nabla$. It makes it possible to obtain a 2-by-2 ODE system for u_x and ρ along particle paths — the so-called Lagrangian formulation. Phase plane analysis is then employed to study the finiteness of the ODE solutions and therefore C^1 regularity of the PDE solution. Similar results stay valid for Euler–Poisson systems with geometric symmetry in higher dimensions [3,8]. To treat genuinely multi-D cases, Liu and Tadmor introduce in [8] the method of spectral dynamics which relies on the ODE system governing eigenvalues of

$$M := \nabla\mathbf{u},$$

which is the velocity gradient matrix, along particle paths. They identify if-and-only-if, pointwise conditions for global existence of C^1 solutions to *restricted* Euler–Poisson systems. Chae and Tadmor [10] further extend the Critical Threshold argument to multi-D full Euler–Poisson systems (1.2a), (1.2b) with attractive forcing $k<0$. Their result, however, offers a blow-up region $\nabla\times\mathbf{u}_0 = 0$, $\operatorname{div}\mathbf{u}_0 < -\sqrt{-nkc}$ which is only a subset of the blow-up region in (1.4). This subset is to the left of the solid line $d\leq d^- := -\sqrt{-nkc}$ depicted in Fig. 4.1. Finally, a recent paper by Tadmor and Wei [12] reveals the critical threshold phenomena in the 1D Euler–Poisson system with pressure.

When tracking other results on the well-posedness of Euler–Poisson equations, we find them commonly relying on (the vast family of) energy methods and thus fundamentally differ from our pointwise results obtained via the Lagrangian approach. With a repulsive force $k>0$, we refer to [13,14] for the global existence of classical solutions with small data and [15] for the nonexistence of global solutions. With attractive force $k<0$, see [1] for local regularity of classical solutions and [16,17] for nonexistence results. Discussions on weak solutions of Euler–Poisson systems can be found in e.g. [18–20]. We also refer to [21–25] and references therein for steady-state solutions. The study of the Euler–Poisson system with damping relaxation can be found in e.g. [26–28].

The rest of this paper is organized as follows. In Section 2, we follow the idea of [10] to derive along particle paths an ODE system governing the dynamics of eigenvalues for $S := \frac{1}{2}(M+M^\top)$. This is a variation of the spectral dynamics for M introduced in [8]. We then derive in Section 3 a closed 2×2 ODE system (3.1) at the cost of turning one equation into inequality. By the comparison principle, this inequality is in favor of blow-up. Thus, with the inequality sign being replaced with an equality sign, a modified ODE system is used to yield sub-solutions and to study a blow-up scenario for the original system. Section 4, devoted to the modified system, reveals the Critical Threshold for such a system. Consequently, a pointwise blow-up condition for the original system is identified. Finally, in Section 5 we prove Corollary 1.2 regarding the Euler–Poisson system with zero background using techniques developed in previous sections.

2. Spectral dynamics

We examine the gradient matrix $M = \nabla\mathbf{u}$ and its symmetric part, $S = \frac{1}{2}\left(\nabla\mathbf{u}+(\nabla\mathbf{u})^\top\right)$. Both matrices are used to study the spectral dynamics of Euler systems (see e.g. [8] for M and [10] for S). The relation between the spectra of M and S is described in the following.

Proposition 2.1. *Let* $\{\lambda_M\}$ *denote the eigenvalues of* M *and* $\{\lambda_S\}$ *for* S*. Then*

$$\sum_{\lambda_M}\lambda_M = \sum_{\lambda_S}\lambda_S = \operatorname{div}\mathbf{u}, \tag{2.1}$$

$$\sum_{\lambda_M}\lambda_M^2 = \sum_{\lambda_S}\lambda_S^2 - \frac{1}{2}|\omega|^2. \tag{2.2}$$

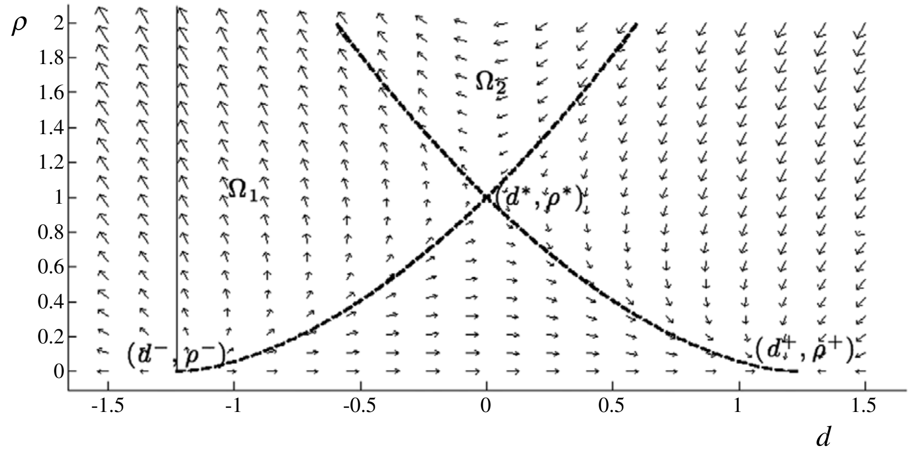

Fig. 4.1. Phase plane of (4.1) with blow-up region $\Omega_1 \cup \Omega_2$ which extends the Chae–Tadmor region [10] $d \le d^-$.

Here, ω *is the* $\frac{n(n-1)}{2}$ vorticity *vector which consists of the off-diagonal entries of* $A := \frac{1}{2}\left(\nabla\mathbf{u} - (\nabla\mathbf{u})^\top\right)$.

Proof. Use identity $M = S + A$ and the skew-symmetry of A,

$$\sum_{\lambda_M} \lambda_M = \mathrm{tr}(M) = \mathrm{tr}(S + A) = \mathrm{tr}(S) = \sum_{\lambda_S} \lambda_S.$$

Squaring the last identity we have $M^2 = S^2 + A^2 + AS + SA$ and therefore,

$$\sum_{\lambda_M} \lambda_M^2 = \mathrm{tr}(M^2) = \mathrm{tr}(S^2 + A^2 + AS + SA) = \sum_{\lambda_S} \lambda_S^2 + \mathrm{tr}(A^2).$$

Note that $AS + SA$ is skew-symmetric and thus traceless. A simple calculation yields $\mathrm{tr}(A^2) = -\frac{1}{2}|\omega|^2$. □

Following [8], we turn to study the dynamics of M along particle paths. Take the gradient of (1.2b) to find

$$M' + M^2 \equiv M_t + u \cdot \nabla M + M^2 = -R(\rho - 1), \tag{2.3}$$

where R stands for the *Riesz matrix*, $R = \{R_{ij}\} := \{\partial_{x_i x_j} \Delta^{-1}\}$.

The trace of (2.3) then yields that the divergence, $d := \mathrm{tr}(M)$, is governed by

$$d' = -\sum_{\lambda_M} \lambda_M^2 - (\rho - 1),$$

and in view of (2.2),

$$d' = -\sum_{\lambda_S} \lambda_S^2 + \frac{1}{2}|\omega|^2 - (\rho - 1). \tag{2.4}$$

We now make the first observation regarding the invariance of the vorticity ω: taking the skew-symmetric part of the M- equation (2.3),

$$A' + AS + SA = 0. \tag{2.5}$$

It follows that if the initial vorticity vanishes, $\omega_0(\bar{x}) \mapsto \nabla \times \mathbf{u}_0(\bar{x}) = 0$, then by (2.5), $\omega \mapsto \nabla \times \mathbf{u}$ vanishes along the particle path which emanates from $\bar{x}$. This allows us to decouple the vorticity and divergence dynamics, and (2.4) implies

$$d' = -\sum_{\lambda_S} \lambda_S^2 - (\rho - 1), \qquad \nabla \times \mathbf{u} = 0. \tag{2.6}$$

Finally, we use Cauchy–Schwartz $\sum \lambda_S^2 \ge \frac{1}{n}\left(\sum \lambda_S\right)^2 = \frac{1}{n}d^2$ and the fact that all λ_S are real (due to the symmetry of S), to deduce the *inequality*,

$$d' \le -\frac{1}{n}d^2 - (\rho - 1). \tag{2.7a}$$

This, together with the mass equation (1.2a) which can be written along particle path

$$\rho' = -d\rho, \tag{2.7b}$$

give us the desired closed system which dominates (ρ, d) along particle paths.

Remark 2.1. The approach pursued in this paper will be based on the *inequality* (2.7a) and is therefore limited to derivation of a finite time breakdown. To argue the global regularity, one needs to study the underlying *equality* (2.6), and to this end, to study the trace $\sum \lambda_S^2$. In the two-dimensional case, for example, one can use $\sum \lambda_S^2 = d^2/2 + \eta^2/2$ to replace (2.7a) with

$$d' = -\frac{1}{2}d^2 - \frac{1}{2}\eta^2 - (\rho - 1), \qquad \eta := \lambda_{S,2} - \lambda_{S,1}.$$

In this framework, global 2D regularity is dictated by the dynamics of the *spectral gap*, $\eta = \lambda_{S,2} - \lambda_{S,1}$, which in turn requires the dynamics of the Riesz transform $R(\rho - 1)$.

3. A comparison principle with a majorant system

The blow-up analysis, driven by the inequalities (2.7),

$$d' \le -\tfrac{1}{n}d^2 - (\rho - 1), \tag{3.1a}$$

$$\rho' = -d\rho. \tag{3.1b}$$

is carried out by standard comparison with the majorant system

$$e' = -\tfrac{1}{n}e^2 - (\zeta - 1), \tag{3.2a}$$

$$\zeta' = -e\zeta. \tag{3.2b}$$

The following proposition guarantees the monotonicity of the solution operator associated with (3.1).

Lemma 3.1. *The following monotone relation between system* (3.1) *and system* (3.2) *is invariant forward in time,*

$$\begin{cases} d(0) < e(0) \\ 0 < \zeta(0) < \rho(0) \end{cases} \text{implies} \begin{cases} d(t) < e(t) \\ 0 < \zeta(t) < \rho(t) \end{cases} \text{for } t \ge 0, \tag{3.3}$$

as long as all solutions remain finite on the time interval $[0, t]$.

Proof. Invariance of positivity of ζ is a direct consequence of (3.2b) and finiteness of e. The rest can be proved by contradiction. Suppose t_1 is the earliest time when (3.3) is violated. Then,

$$\zeta(t_1) = \zeta(0)\exp\left(-\int_0^{t_1} e(t)\mathrm{d}t\right) < \rho(0)\exp\left(-\int_0^{t_1} d(t)\mathrm{d}t\right) = \rho(t_1). \tag{3.4}$$

Therefore, we are left with only one possibility, namely, $e(t_1) = d(t_1)$. Subtracting (3.1a) from (3.2a),

$$(e-d)' \geq -\frac{1}{n}(e^2-d^2) - (\zeta - \rho), \tag{3.5}$$

and by (3.4), we find that at $t = t_1$,

RHS of $(3.5)_{|t=t_1} = 0 - [\zeta(t_1) - \rho(t_1)] > 0$.

However, this contradicts the negativity of the expression on the left of (3.5), since $e(t) - d(t) > 0$ for all $t < t_1$ and vanishes at $t = t_1$ which imply that

LHS of $(3.5)_{|t=t_1} = \big(e(t_1) - d(t_1)\big)' \leq 0$. □

In the next section, we employ phase plane analysis on the modified system (3.2). When translated in terms of the original system (3.1), however, such analysis can only yield blow-up results and is insufficient for global existence results. In other words, estimate (3.3) is only useful for proving $d \searrow -\infty$, the key mechanism for blow-up of C^1 solutions.

4. Stability analysis of the majorant system

We shall prove the blow-up of the majorant system (3.2), $e(t) \to -\infty$ as $t \uparrow t_c$, which in turn, by Lemma 3.1 implies $d(t) \to -\infty$. Abusing notations, we express the majorant system in terms of the original variables $(e, \zeta) \mapsto (d, \rho)$:

$$d' = -\frac{1}{n}d^2 - (\rho - 1), \tag{4.1a}$$

$$\rho' = -d\rho. \tag{4.1b}$$

The (in-)stability analysis of (4.1) hinges on the path invariants of this system. To this end, we use the same q-transformation employed in [29,9]: setting $q := d^2$ and differentiate along the path $\{(t, X(a,t)) \mid X_t(a,t) = u(t, X(a,t)), X(a,0) = a\}$, we find

$$\frac{\mathrm{d}q}{\mathrm{d}\rho} = 2d\frac{d'}{\rho'} = \frac{2}{n\rho}q + 2\left(1 - \frac{1}{\rho}\right),$$

which yields

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\left(q\rho^{-\frac{2}{n}}\right) = 2(1-\rho^{-1})\rho^{-\frac{2}{n}}.$$

Upon integration, we arrive at the following key observation.

Lemma 4.1. *The majorant system (4.1) is equipped with the path invariant,*

$I(d(t), \rho(t)) = I(d_0, \rho_0)$,

along each path $(t, x(t))$ initiated with a non-vacuum state $(d_0, \rho_0 > 0)$. Here,

$$I(d,\rho) := d^2\rho^{-\frac{2}{n}} - 2\int_1^\rho (1-r^{-1})r^{-\frac{2}{n}}\,\mathrm{d}r = \rho^{-\frac{2}{n}}\left(d^2 - nF(\rho)\right), \tag{4.2}$$

where $F(\cdot)$ is specified in (1.3b).

It is a simple calculation to show that the majorant system (4.1) admits three distinct critical points (see Fig. 4.1):

$(d^*, \rho^*) := (0, 1), \quad (d^\pm, \rho^\pm) := (\pm\sqrt{n}, 0)$.

and that $(0, 1)$ is a saddle point, $(-\sqrt{n}, 0)$ a nodal source and $(\sqrt{n}, 0)$ a nodal sink. The separatrix is given by the zero level set $I(d, \rho) = 0$. Moreover, the right branch of the separatrix, $d = \sqrt{nF(\rho)}$ connects critical points $(0, 1)$ and $(\sqrt{n}, 0)$ while the left branch, $d = -\sqrt{nF(\rho)}$ connects $(0, 1)$ and $(-\sqrt{n}, 0)$.

By inspection of the phase plane in Fig. 4.1, we postulate the following invariant region of finite-time blow-up for the modified system (4.1),

$$\Omega = \Omega_1 \cup \Omega_2 = \{(d,\rho) \mid d < \operatorname{sgn}(\rho - 1)\sqrt{nF(\rho)}\} \tag{4.3a}$$

where

$$\Omega_1 := \{(d,\rho) \mid I(d,\rho) > 0 \text{ and } d < 0 \text{ and } \rho > 0\}, \tag{4.3b}$$

$$\Omega_2 := \{(d,\rho) \mid I(d,\rho) < 0 \text{ and } \rho > 1\}. \tag{4.3c}$$

Lemma 4.2. *Consider the modified system (4.1), equipped with initial data (d_0, ρ_0). If $(d_0, \rho_0) \in \Omega$, then $\operatorname{div}\mathbf{u} \to -\infty$ and $\rho \to \infty$ at a finite time.*

Proof. We begin by recalling (1.3b), consult (4.2),

$$F(\rho) = \frac{2}{n}\rho^{\frac{2}{n}}\int_1^\rho (1-r^{-1})r^{-\frac{2}{n}}\,\mathrm{d}r.$$

Clearly, $F(1) = F'(1) = 0$ and a simple calculation shows that $F''(\rho) = \frac{2}{n}\rho^{\frac{2}{n}-2}$, which implies that $F(\rho)$ is a strictly convex function of positive ρ and attains its only minimum at $\rho = 1$,

$$F(\rho) \geq F(1) = 0. \tag{4.4}$$

We shall also utilize the invariance of (4.2)

$$d^2 - nF(\rho) = \rho^{\frac{2}{n}}I_0, \qquad I_0 = I(d_0, \rho_0). \tag{4.5}$$

We now turn to discuss the two possible blow-up scenarios, depending whether the initial data (d_0, ρ_0) belong to the blow-up regions Ω_1 or Ω_2 given in (4.3).

Case #1. Assume that $(d_0, \rho_0) \in \Omega_1$ so that the invariant I remains a *positive* constant

$I > 0$.

In this case, d remains negative, for otherwise, setting $d = 0$ in (4.5) would result in $F(\rho) = -\rho^{\frac{2}{n}}I/n < 0$, violating (4.4). Thus, (4.5) and (4.4) yield an upper bound,

$d \leq -\rho^{\frac{1}{n}}\sqrt{I}$.

Then, by (4.1b), we have a Riccati type of equation $\rho' \geq \sqrt{I}\rho^{1+\frac{1}{n}}$ for which the solution exhibits blow-up $\rho \to +\infty$ and the divergence $d = \operatorname{div}\mathbf{u}$ approaches $-\infty$ at a finite time due to (4.5).

Case #2. Assume that $(d_0, \rho_0) \in \Omega_2$ so that the invariant I remains a *negative* constant

$I < 0$.

In this case, $\rho - 1$ remains positive, for otherwise setting $\rho = 1$ in (4.5) would result in $F(1) = (d^2 - I)/n > 0$ in contradiction to (4.4). Now, for $\rho > 1$ we have

$$F(\rho) = \frac{2}{n}\rho^{2/n}\int_1^\rho \left(1 - \frac{1}{r}\right)\frac{1}{r^{2/n}}\mathrm{d}r \leq \frac{2}{n}\rho^{2/n}(\rho - 1).$$

This together with (4.5) yield

$$\frac{2}{n}\rho^{2/n}(\rho-1) \geq F(\rho) = \frac{1}{n}\left(d^2 - \rho^{2/n}I\right) \geq -\frac{1}{n}\rho^{2/n}I$$

and the lower bound, $\rho - 1 \geq -I/2$ follows. Thus, by (4.1a), we end up with a Riccati type of equation

$$d' \leq -\frac{d^2}{n} + \frac{I}{2}.$$

Since the invariant I remains a negative constant, the solution exhibits blow-up $d = \operatorname{div}\mathbf{u} \to -\infty$ at a finite time even if initially $d_0 > 0$. The density ρ also approaches ∞ in finite time due to (4.5). □

The last step of proving the Main theorem is just to combine the comparison principle in Lemma 3.1 with the above lemma. We notice that Ω is an open set and thus given any initial data $(d_0, \rho_0) \in \Omega$ for the original system, we can always find $\varepsilon > 0$ and initial data $(d_0 + \varepsilon, \rho_0 - \varepsilon) \in \Omega$ for the modified system. This latter initial data will lead to a finite time blow-up of the modified system and therefore, by Lemma 3.1, initial data $(d_0, \rho_0) \in \Omega$ will lead to finite time blow-up of the original system.

5. Critical threshold for zero background

We now turn to the attractive Euler–Poisson system (1.1a), (1.1b) with zero background $c = 0$ and prove Corollary 1.2. For simplicity, we only show the case with $k = -1$ since a straightforward rescaling argument, $x \to \sqrt{-k}\,x$ and $t \to \sqrt{-k}\,t$, will cover the case for general $k < 0$.

Proof of Corollary 1.2. Following the same calculation that leads to the majorant system (4.1a), (4.1b), we arrive at a similar ODE system for the case $c = 0$, $k = -1$,

$$d' = -\frac{1}{n}d^2 - \rho, \tag{5.1a}$$

$$\rho' = -d\rho. \tag{5.1b}$$

Then, as an analogue to the invariant (4.2), we find the corresponding invariant,

$$I(d, \rho) := d^2\rho^{-\frac{2}{n}} - 2\int_a^\rho r^{-\frac{2}{n}}\,\mathrm{d}r.$$

By choosing the constant

$$a = \begin{cases} +\infty, & n = 1, \\ 1, & n = 2, \\ 0, & n \geq 3, \end{cases}$$

we have

$$I(d, \rho) = \begin{cases} d^2\rho^{-2} + 2\rho^{-1}, & n = 1 \\ d^2\rho^{-1} - 2\ln\rho, & n = 2 \\ d^2\rho^{-\frac{2}{n}} - \dfrac{2n}{n-2}\rho^{1-\frac{2}{n}}, & n \geq 3. \end{cases} \tag{5.2}$$

Using the positivity of ρ and d^2 in (5.2), we have $I \geq 2\rho^{-1} > 0$ for $n = 1$ and $I \geq -2\ln\rho$ for $n = 2$. Both estimates imply that ρ is bounded from below by a positive constant. In the case of $n \geq 3$, the sup-critical condition (1.5) implies $I < 0$. Thus, by (5.2), we have $0 > I \geq -\frac{2n}{n-2}\rho^{1-\frac{2}{n}}$ which, again, implies ρ is greater than a positive constant.

Therefore, by (5.1a), d satisfies a differential inequality

$$d' \leq -\frac{d^2 + \alpha}{n}$$

with positive constant α. Obviously, $d(t)$ approaches $-\infty$ at a time no later than $\frac{n\pi}{2\sqrt{\alpha}}$. $\square$

Acknowledgment

Research was supported in part by NSF grants 07-07949, 07-57227 and ONR grant N000140910385.

References

[1] Tetu Makino, On a local existence theorem for the evolution equation of gaseous stars, in: Patterns and Waves, in: Stud. Math. Appl., vol. 18, North-Holland, Amsterdam, 1986, pp. 459–479.

[2] Uwe Brauer, Alan Rendall, Oscar Reula, The cosmic no-hair theorem and the non-linear stability of homogeneous Newtonian cosmological models, Classical Quantum Gravity 11 (9) (1994) 2283–2296.

[3] Michael P. Brenner, Thomas P. Witelski, On spherically symmetric gravitational collapse, J. Statist. Phys. 93 (3-4) (1998) 863–899.

[4] Yinbin Deng, Tai-Ping Liu, Tong Yang, Zheng-an Yao, Solutions of Euler-Poisson equations for gaseous stars, Arch. Ration. Mech. Anal. 164 (3) (2002) 261–285.

[5] Lee Hartmann, Accretion processes in star formation, in: Cambridge Astrophysics Series, vol. 32, Cambridge University Press, 2009.

[6] Grégoire Loeper, Quasi-neutral limit of the Euler-Poisson and Euler-Monge-Ampère systems, Comm. Partial Differential Equations 30 (7–9) (2005) 1141–1167.

[7] Shlomo Engelberg, Hailiang Liu, Eitan Tadmor, Critical thresholds in Euler-Poisson equations, Indiana Univ. Math. J. 50 (Special Issue) (2001) 109–157. Dedicated to Professors Ciprian Foias and Roger Temam (Bloomington, IN, 2000).

[8] Hailiang Liu, Eitan Tadmor, Spectral dynamics of the velocity gradient field in restricted flows, Comm. Math. Phys. 228 (3) (2002) 435–466.

[9] Hailiang Liu, Eitan Tadmor, Critical thresholds in 2D restricted Euler-Poisson equations, SIAM J. Appl. Math. 63 (6) (2003) 1889–1910. (electronic).

[10] Dongho Chae, Eitan Tadmor, On the finite time blow-up of the Euler–Poisson equations in R^2, Commun. Math. Sci. 6 (3) (2008) 785–789.

[11] Hailiang Liu, Eitan Tadmor, Critical thresholds and conditional stability for Euler equations and related models, in: Hyperbolic Problems: Theory, Numerics, Applications, Springer, Berlin, 2003, pp. 227–240.

[12] Eitan Tadmor, Dongming Wei, On the global regularity of subcritical Euler-Poisson equations with pressure, J. Eur. Math. Soc. (JEMS) 10 (3) (2008) 757–769.

[13] Yan Guo, Smooth irrotational flows in the large to the Euler-Poisson system in $\mathbf{R}^{3+1}$, Comm. Math. Phys. 195 (2) (1998) 249–265.

[14] Stéphane Cordier, Emmanuel Grenier, Quasineutral limit of an Euler-Poisson system arising from plasma physics, Comm. Partial Differential Equations 25 (5–6) (2000) 1099–1113.

[15] Benoît Perthame, Nonexistence of global solutions to Euler-Poisson equations for repulsive forces, Japan J. Appl. Math. 7 (2) (1990) 363–367.

[16] Tetu Makino, Blowing up solutions of the Euler-Poisson equation for the evolution of gaseous stars, in: Proceedings of the Fourth International Workshop on Mathematical Aspects of Fluid and Plasma Dynamics, Kyoto, 1991, vol. 21, 1992, pp. 615–624.

[17] Tetu Makino, Benoît Perthame, Sur les solutions à symétrie sphérique de l'équation d'Euler-Poisson pour l'évolution d'étoiles gazeuzes, Japan J. Appl. Math. 7 (1) (1990) 165–170.

[18] Bo Zhang, Global existence and asymptotic stability to the full 1D hydrodynamic model for semiconductor devices, Indiana Univ. Math. J. 44 (3) (1995) 971–1005.

[19] Pierangelo Marcati, Roberto Natalini, Weak solutions to a hydrodynamic model for semiconductors and relaxation to the drift-diffusion equation, Arch. Rational Mech. Anal. 129 (2) (1995) 129–145.

[20] F. Poupaud, M. Rascle, J.-P. Vila, Global solutions to the isothermal Euler-Poisson system with arbitrarily large data, J. Differential Equations 123 (1) (1995) 93–121.

[21] Irene Martínez Gamba, Stationary transonic solutions of a one-dimensional hydrodynamic model for semiconductors, Comm. Partial Differential Equations 17 (3–4) (1992) 553–577.

[22] Pierre Degond, Peter A. Markowich, A steady state potential flow model for semiconductors, Ann. Mat. Pura Appl. 165 (4) (1993) 87–98.

[23] T. Luo, J. Smoller, Nonlinear dynamical stability of newtonian rotating and non-rotating white dwarfs and rotating supermassive stars, Comm. Math. Phys. (2008) 166–+.

[24] Tao Luo, Joel Smoller, Rotating fluids with self-gravitation in bounded domains, Arch. Ration. Mech. Anal. 173 (3) (2004) 345–377.

[25] Gerhard Rein, Non-linear stability of gaseous stars, Arch. Ration. Mech. Anal. 168 (2) (2003) 115–130.

[26] Dehua Wang, Global solutions and relaxation limits of Euler-Poisson equations, Z. Angew. Math. Phys. 52 (4) (2001) 620–630.

[27] Dehua Wang, Gui-Qiang Chen, Formation of singularities in compressible Euler-Poisson fluids with heat diffusion and damping relaxation, J. Differential Equations 144 (1) (1998) 44–65.

[28] Tao Luo, Roberto Natalini, Zhouping Xin, Large time behavior of the solutions to a hydrodynamic model for semiconductors, SIAM J. Appl. Math. 59 (3) (1999) 810–830 (electronic).

[29] Hailiang Liu, Eitan Tadmor, Rotation prevents finite-time breakdown, Phys. D 188 (3-4) (2004) 262–276.